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RADICAL ANTI-REALISM AND SUBSTRUCTURAL LOGICS

1. INTRODUCTION

According to the realist, the meaning of a declarative, non-indexical sentence is the condition under which it is true and the truth-condition of an undecidable sentence can obtain or fail to obtain independently of our capacity, even in principle, to recognize that it obtains or that fails to do so.[1] In a series of papers, beginning with 'Truth' in 1959, Michael Dummett challenged the position that the classical notion of truth-condition occupied as the central notion of a theory of meaning, and proposed that it should be replaced by the anti-realist (and intuitionistic) notion of assertability-condition. Taken together with work from Dag Prawitz, Dummett's work truly opened up the anti-realist challenge at the level of proof-theoretical semantics.[2] There has been since numerous rejoinders from partisans of classical logic, which were at times met with by attempts at watering down the anti-realist challenge, e.g., by arguing that anti-realism does not necessarily entail the adoption of intuitionistic logic. Only a few anti-realists, such as Crispin Wright and Neil Tennant, tried to look instead in the other direction, towards a more radical version of anti-realism which would entail deeper revisions of classical logic than those recommended by intuitionists.[3] In this paper, which is largely programmatic, we shall also argue in favour of a radical anti-realism which would be a genuine alternative to the traditional anti-realism of Dummett and Prawitz. The debate about anti-realism has by now more or less run out of breath and we wish to provide it with a new lease on life, by taking into account the profound changes that took place in proof theory during the intervening years. We have in mind in particular the considerable development within Gentzen-style proof theory of non-classical, substructural logics other than intuitionistic logic, which seriously opens up the possibility that anti-realism, when properly understood, might end up justifying another logic, and the development of closer links between proof theory and computational complexity theory that has renewed interest in a radical form of anti-realism, namely strict finitism.

We shall first provide the outline of an argument in favour of a radical form of anti-realism premised on the need to comply with two principles, implicitness and immanence, when trying to frame assertability-conditions. It follows from the first principle that one ought to avoid explicit bounding of the length of computations, as is the case for some strict finitists, and look for structural weakening instead. In order to comply with the principle of immanence, one ought to take into account the difference between being able to recognize a proof when presented with one and being able to produce one and thus avoid the idealization of our cognitive capacities that arise within Hilbert-style calculi. We then explore the possibility of weakening structural rules in order to comply with radical anti-realist strictures.

2. FROM TRADITIONAL TO RADICAL ANTI-REALISM

It has been granted by Dummett himself, in his 1972 postscript to 'Truth', that his original challenge against the notion of truth-conditions is easily met by the realist (1978, 23-4). The realist wishes not to identify what makes a sentence true or false with that by which we recognize it as true or as false but the anti-realist argues that understanding of a statement consists of the knowledge of 'recognizable circumstances' that determine it as true or false. Thus, the anti-realist finds the knowledge of the conditions under which a sentence is, say, true, when that condition is not one which is always capable of being recognized as obtaining, to be a rather obscure notion. It is open to the realist to agree with the anti-realist that truth-conditions should not, in that sense, be transcendent. However, in order for the realist to recover her position it suffices that she claims by the same token that any given truth may be recognized by an hypothetical being whose cognitive capacities may exceed our limited ones. So, the difference between the anti-realist and the realist reduces to this crucial point: the former interprets 'capable of being known' as 'capable of being know by us', the latter interprets it as 'capable of being known by some hypothetical being whose intellectual capacities and powers of observation may exceed our own' (Dummett 1978, p.24). It follows from this that, in order to have a veritable case against realism, the anti-realist should avoid defining assertability-conditions in terms of the cognitive abilities of an hypothetical being and define them in terms of our own, limited human cognitive capacities.

Any definition of assertability-conditions along those lines should satisfy two principles, namely that, first, conditions under which the assertion of a sentence is justified must, when satisfied, always be recognized as being satisfied and, secondly, that the definition of the conditions under which the assertion of a sentence is justified must, as is the case with classical definitions of truth-conditions, avoid making an *explicit* appeal to our cognitive capacities. We shall call these, respectively, the principles of 'immanence' and 'implicitness'. The former derives from the very nature of the anti-realist challenge against realism: if it were not always the case that, when satisfied, assertability-conditions are recognized as being so, then the anti-realist position would allow for 'transcendent' conditions and would simply not vary significantly from a realist position. The *raison d'être* of the principle of implicitness should also be obvious. If we wish assertability-conditions to play the role played by truth-conditions, which do not contain any reference to psychological content, then we must abstain from defining assertability in terms that would presuppose that the nature of our cognitive capacities is known and thus analyse these capacities in a neutral, non-indexical language in which they do not appear as ours. As is already the case with truth-conditions, which they are supposed to replace, assertability-conditions must therefore be *formally* defined, that is that it must be possible unambiguously to indicate the conditions under which we are automatically justified to assert a complex statement in function of the conditions justifying the assertion of its immediate constituents: it should not be necessary to wonder, on the occasion of each application of one of the clauses of this recursive rule, if the assertion to which we are entitled to by this rule really is the assertion of a statement that our own cognitive capacities would, alone, allow to recognize as true.

The principle of implicitness allows us to rule out some candidates for assertability-conditions. We have in mind here strict finitist variants that could be developed along lines first propounded by A. S. Esenin-Volpin (1961, 1970). The term 'strict finitism' seems to cover two conflicting theses about the central notion of 'feasibility': *Gandy's Thesis*, according to which there is an upper bound B, independent of time, to the size of mathematical inscriptions that can be considered by mathematicians, in particular, if $n > 2^{B}$, then the numeral for n cannot be concretely presented, i.e., it is not feasible; and the *Karp-Cook Thesis* according to which a set of strings is feasibly computable if and only if it is polynomial-time computable.[4] This approach to feasible computing is linked rather with a finitist or Aristotelian ontology of infinities that are conceived of as unbounded processes as opposed to totalities, and it thus clashes with rejection of large finite numbers by those adhering strictly to Gandy's Thesis.[5]

The traditional strict finitist critique, in Gandy's Thesis, calls for an explicit bounding of human cognitive capacities which violates our principle of implicitness. In 'Wang's Paradox', Dummett pointed out that the vagueness of predicates such as 'feasible' renders them susceptible to a variant of the Sorites. For Dummett concepts such 'feasible' are thus semantically incoherent and strict finitism, which admits of them, is not viable as a philosophy of mathematics (1978, p. 265). In defense of strict finitism, one could point out, however, that Dummett's strategy has a notorious flaw, first pointed out by Crispin Wright in his paper on 'Strict Finitism' (1993), in this that the presumed incoherence of strict finitist notions is also a problem for intuitionism. Instead of rehearsing Wright's arguments we should like simply to point out that Rohit Parikh has already shown in 'Existence and Feasibility' (1971) that, while the system resulting of the addition to Peano arithmetic of the predicate 'feasible' is inconsistent, all theorems proved in it whose proofs are 'short' are in fact true.[6] A proof of contradiction Hilbert-style, i.e., of something like $0 = 1$, would be itself of non-feasible length and could not be recognized by anyone consistently adhering to the strict finitist viewpoint. This result already goes a long way towards legitimizing the notion of 'feasibility' and Dummett's argument loses much of its force. It still does not make the resulting system more palatable for foundational purposes; it remains, however, of interest for the analysis of computation. The morale that we would like to draw is that in order to avoid violating the principle of implicitness *bounds should remain hidden*, i.e., the logic for radical anti-realism should reflect limitations to human cognitive capacities in a 'structural' fashion. It cannot be the simple matter of, say, adopting intuitionistic logic and bounding the length of computations within it, as strict finitism is often taken to be. So, in defending a radical anti-realist stance, we are emphatically not propounding a variant of Gandy's Thesis.

To come back to the principle of immanence. The traditional definition of assertability-conditions for mathematical statements reads something like this: a statement is assertable if there exists an effective proof of it, that is a finite sequence of statements of which it is the last and of which every statement follow another as the result of an application of a rule of inference —there are, of course, only a finite number of such rules. Such as definition fully satisfies the principle of implicitness but only partially the principle of immanence. It satisfies it because it does not allow for an hypothetical being whose cognitive capacities would be such that it could, say, recognize the truth of a universal statement by inspection of an infinity of particular cases. The realist could still point out, however, that when the anti-realist admits of finite proofs that can be carried out merely in principle he does not fare much better than someone who admits of truth-conditions which transcend our cognitive capacities. Therefore, in order for the definition fully to satisfy the principle of immanence, our cognitive capacities must allow us *always* to recognize a sentence as assertable when it is, that is that one must be able to recognize an object Pr(*s*) which is an effective proof of *s*, when there is one. As is well-known, this is ambiguous, since one may understand this either, as Dummett did, as the *weaker* claim that one has to be able recognize a proof of *s* when presented with one or, as we would argue, as the *stronger* claim that one must be able to produce or reproduce the object Pr(*s*).

An argument in favour of this stronger claim can be made along the following line:[7] for the anti-realist really to distinguish his position from that of the realist on this rather crucial point, he must claim not only that circumstances in which an assertion is justified must be such that we should recognize them when we are in a position to do so, he must also claim that *we must always be able in practice to put ourselves in such a position* whenever such circumstances exist. Otherwise, it would be open for the realist to admit there should always exist circumstances under which we would recognize that an assertion is justified and merely to deny that we should always have the practical capacity to put ourselves in that position. To repeat, the *weaker* claim that one has to be able recognize a proof of *s* when presented with one won't do, because there may simply be situations where we could recognize a proof when presented with one, but we would never be able in practice to put ourselves in such a position. Therefore, in order to develop a coherent alternative to the realist, the anti-realist must develop a notion of assertability-conditions based on the fact that our own cognitive capacities must allow us not only *always* to recognize a sentence as assertable when it is, that is that one must be able to recognize the object Pr(*s*) which is an effective proof of *s*, when there is one, but also to be able in practice to produce or construct the object Pr(*s*).

3. TWO CONCEPTS OF PROOF

To give a more precise content to our last claim, we propose that one distinguishes between two different notions of proof, namely those of proof as 'object' and as 'act'. According to the first conception, a proof is something like an assemblage of strings of symbols satisfying such and such property. From the second, more dynamic, conception, a proof is a process whose result may be represented or described by means of linguistic symbols. Within the framework of Hilbert-style calculi, a proof of *s* is typically conceived , in accordance with the first conception, as an object, namely as a series of well-formed formulas terminating with *s* and obeying to simple, decidable properties. We shall argue that this conception is invalidated by radical anti-realist strictures.

The main virtue of Hilbert-style concept of proof lies in their effectiveness. This property insures that the proofs may be *ratified*. As Church once remarked (1956, 52-3), the existence of a routine procedure allowing to decide whether a given string of symbols is in conformity or not with given rules is required as a guaranty for the control and the communication of assertions within the mathematical community. By putting this property to the fore, one is unavoidably driven to the parasitic idea that our activity might be reduced to that of control or ratification, which does not require any particular cognitive resources. If the only property of the proof predicate is decidability in principle, then the very same abilities that are requested to *ratify* the proofs could easily be conceived as being also sufficient to *produce* them. For the simple capacity to decide *in principle* if a given sequence of strings is a genuine proof or not guarantees also that we are able to enumerate the theorems, e.g., by lexicographically enumerating the set of the sequences of well-formed strings and by applying to each of them a suitable test of conformity. Thus, the restriction to the effective methods of proof, initially intended as a guaranty that the proofs can be *ratified*, may eventually be invoked to claim that we are able to *produce* them at the same (cognitive) cost. To sum up, the mere reference to the decidability in principle of the concept of proof does not really allow one to draw a frontier between two activities whose difference is, according to us, of paramount importance, namely between *ratifying* and *producing*. To draw such a frontier, we have to substitute feasibility for effectivity in principle; this very distinction goes, as far as proofs are concerned, far beyond the framework of the Hilbert-style calculi.

It is not difficult to show that the limitation to the mere requirement of effectivity in principle is at the very root of the so-called ‘Platonism’ of proofs. An effective proof in an Hilbert-style calculus may ‘exist’ even if any one equipped with human cognitive capacities (or any extension of them such as a computer) could not be in a position to produce it within a reasonable time. Therefore, one cannot infer from the fact that some one is in practice in a position to *recognize* a proof in an Hilbert-style calculus when presented with one that that person is in practice in a position to *produce* it, even when it is in principle possible. *The existence of a proof Pr(s) in an Hilbert-style calculus is therefore too large a criterion for the assertability of s*. Such calculi make room only the *ratification* of proofs, i.e., for the possibility of recognizing a proof when presented with one, but not for the possibility, by any one equipped with real human cognitive capacities, of *producing* it.

The conception of proofs that underlies Hilbert-style calculi is such that consequences of the applications of the rules of inference are not inferred by us but they already infer themselves, so to speak. Our task is accordingly not one of inferring the consequences of the hypotheses, in accordance with the rules of inference: it is merely one of *ratifying* an already written out but possibly hitherto never read proof.[8] By identifying proofs to objects independent of us, our activity has been reduced to that of control or ratification, which does not require any particular cognitive resources. *It is precisely in this context that one forges an idealized picture our cognitive capacities*.

Another conception of proofs has its roots in the Brouwer-Heyting-Kolmogorov (BHK) interpretation of logical connectives, which consists of defining the *act* justifying the assertion of a statement in terms of the acts justifying the assertion of its immediate constituents. Thus, to take only two examples, the act justifying A&B is given by the act justifying the assertion of A and by the act justifying the assertion of B, and the act justifying $A \rightarrow B$ is given by a construction transforming any act justifying the assertion of A into an act justifying the assertion of B. *This is a conception of proofs as acts, not as registers of an independent reality*. It has been validated by the introduction by Gentzen (1969) of natural deduction and sequent calculi and it has been developed within the current paradigm of 'propositions as types', in particular by Per Martin-Löf:

> A proof is, not an object, but an act. This is what Brouwer wanted to stress by saying that a proof is a mental construction, because what is mental, psychic, is precisely our acts, and the word construction, as used by Brouwer, is but a synonym for proof. Thus he might just as well have said that the proof of a judgement is the act of proving, or grasping, it. And the proof is primarily the act as it is being performed. Only secondarily, and irrevocably, does it become the act that has been performed. (Martin-Löf 1985, p. 231)

We believe that Wittgenstein was the only philosopher in the 20th century who tried consistently to develop a non-Platonist conception of rules and proofs. His ideas, although dated and often obscurely put, can still be of interest, especially in the current context.[9] Wittgenstein has been heavily criticized for having said in *Remarks on the Foundations of Mathematics* (1978) that one must be able to produce or reproduce the object which is the proof of a given assertion (1978, III, § 1), but his claims amount to nothing more than the radical anti-realist view just argued for. According to Wittgenstein, a mathematical proof is not an experiment and it must convince us to adopt its conclusion; reproducibility and surveyability are properties of proofs linked with this capacity to convince (1978, III, § 55). Understanding a proof means to be able to take it in (1978 I, § 80 & III, § 9), to be able to reproduce it, etc. In essence, Wittgenstein is claiming that to understand a proof means more than the mere capacity to ratify it. Many salient aspects of the later Wittgenstein's philosophy of mathematics are also put in proper perspective within the confines of this debate. We have in mind not only the argument about the surveyability of proofs —see, e.g., his (1978, III, §§ 2, 3, 14, 16, 18, 19, 43)—, which is linked, for obvious reasons, with the requirement that one must be able to produce or reproduce proofs but also his controversial 'rule-following' argument (Wittgenstein), which is aimed at the conception of 'rules as rails' —as in, e.g., (1953, §§ 218-219)—, which, in turn, characterizes very well the Platonist conception underlying Hilbert-style calculi. There is no space for a discussion of these aspects of Wittgenstein's philosophy or the conception of proofs as acts, we leave this topic for a further paper. At this stage, our claim is that one should strive to define assertability-conditions not in terms of the 'static', Platonic conception underlying Hilbert-style calculi but in terms of this 'dynamic' conception of proofs as acts. A thorough switch to the latter should be, according to us, at the very heart of the debate within anti-realism.

We would like further to claim that part of the reasons why the traditional anti-realist has difficulties truly to distinguish his case from that of the realist has to do with the fact that he surreptitiously relies on the conception of proofs that underlies Hilbert-style calculi. Crispin Wright has already pointed out that, indeed, the traditional anti-realist slips into this sort of thinking when having in mind, for example, so-called ‘decidable’ arithmetical predicates. The anti-realist merely requests, in their case, the possession of a guaranty that the proofs can be ratified, in which case, it is assumed that

> we have, as it were, only a spectator’s role to play; that we are capable of so conferring a meaning on the symbols involved in the statement as to decide the statement’s truth, or falsity, ahead of ourselves and independently of whatever verdict we reach on actually doing the computation if we are able to. […] which result is correct in no sense awaits our judgement; when we do the computation, we merely trace out connections to which, by the meanings which we have given to the relevant signs […], we have already committed ourselves. (Wright 1993, p. 144)

It is precisely because it relies implicitly on this Platonic conception of proofs that traditional anti-realism fails to provide strict enough a criteria for assertability. It therefore be criticized and abandoned if one wishes to hold a radical anti-realist stance.

4. AN ANTI-REALIST LOOK AT STRUCTURAL RULES

Dummett’s paper on ‘The Philosophical Basis of Intuitionistic Logic’ opens with the question: “What plausible rationale can there be for repudiating, within mathematical reasoning, the canons of classical logic in favour of those of intuitionistic logic?” (1978, p. 215). We propose that one ask instead: What sort of logic will one end up with, if one adheres to radical anti-realism? It makes sense to ask such a question today, as opposed to the days when Dummett wrote, because there is a growing interest in non-classical logics, while not so many years ago all logics except classical, and perhaps intuitionistic logic, were considered as simply esoteric.

Our arguments have been so far to the effect that, in order to comply with the principle of implicitness, one ought to avoid explicit bounding of the length of computations and look for structural weakening instead and, furthermore, that in order to comply with the principle of immanence, one ought to avoid the idealizations that arise when one is surreptitiously relying on the Platonic conception of proofs that underlies Hilbert-style calculi. A new approach to the above question suggests itself quite naturally in this context, which was first postulated as a thesis by Kosta Dosen, namely that

> Two logical systems are alternative if, and only if, they differ only in their assumptions on structural deductions. (1989, p. 376)

In a nutshell, while Dummett and traditional anti-realists argued for intuitionism by looking at logical rules in natural deduction calculi, we propose that one looks instead at structural rules in sequent calculi. If one looks at sequent formulations of various logics, one notices that the 'logical' rules may be made not to differ from those given for classical logic —in other words the logical rules are 'invariant'— while the 'structural' rules change. Non-classical logics consist of restrictions put on the structural rules of classical logic.[10]

The alternative 'substructural' logics that can be obtained by restricting structural rules are numerous and include intuitionistic logic, relevant logic, BCK logic, linear logic, and the Lambek calculus of syntactic categories. From this 'substructural' point of view, it is only reasonable to ask: Why should the anti-realist peg his case only on intuitionistic logic and not explore these various other avenues? It is quite noticeable that from the point of view of substructural logics, intuitionistic logic differs from classical logic merely by restricting the rule of Thinning on the right from:

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}$$

to

$$\frac{\Gamma \vdash}{\Gamma \vdash A}$$

(This restriction is explained by the requisite that sequents must have only one conclusion.) Again, one might legitimately ask: *Why should one stop here?* By stopping the movement towards further restrictions of the structural rules where he does, the intuitionist will appear to be adopting an *ad hoc* and potentially fatally unstable stance. Much could be said at this stage, we shall, however, merely explore briefly, for reasons of space, the consequences of the second part of our argument, (the rejection of the Platonist conception of proofs) upon the possibility of further restrictions on structural rules.

In talking earlier about Platonic idealizations of our cognitive capacities, we had in mind idealizations that are hidden in what Prawitz called, in *Natural Deduction*, 'improper inference rules' (1965, p. 23), i.e., those rules which do not state how assumptions are to be discharged. For example, the introduction rule for implication stipulates that if B has been inferred from A, it is possible to construct a new deduction whose conclusion will be A→B, which will not depend any more from A, i.e., the hypothesis A will be discharged. The discharge is indicated by use of square brackets:

$$\begin{array}{c}[A]\\ \cdot\\ \cdot\end{array}$$

$$\dfrac{\begin{array}{c}.\\ B\end{array}}{A \to B}\;\to I$$

(Other improper inference rules are $\vee$E, $\forall$I, and $\exists$E.) The tree-like structure of derivations is such that a given derivation may contain many occurrences of a given hypothesis. Now in intuitionistic logic, as embodied in Gentzen's calculus, the discharge of the hypothesis is merely optional. By making the discharge obligatory one obtains a different logic. With the obligatory discharge, which is the hallmark of relevant logic, one cannot prove $B \to A$ from A, i.e., one cannot construct a deduction such as:

$$\dfrac{\dfrac{A}{B \to A}}{A \to (B \to A)}$$

whose conclusion is one of C. I. Lewis' paradoxes of relevance: if a statement is true then it is implied by any other statement. Gentzen himself felt uneasy about what he called the lack of "factual dependence" between B and A. It should be pointed out that there are obvious anti-realist reasons to require the obligatory discharge: the cognitive resources needed for justifying A alone may turn out to be insufficient for the justification of A on the basis of B, since the cognitive resources needed for B may be lacking.

To limit oneself to a single discharge leads to Girard's linear logic (Girard 1987, 1995). Girard's motivation consists in pointing out (1995, 1f.) that *from the perspective of the physical execution of the rule*, if I have \$ 2 and a pack of Gauloises cigarettes costs \$ 1, then the result of buying only one pack of Gauloises is that one lose only \$ 1. In linear logic one cannot get $A \to B$ from $(A \& A) \to B$ because there is no reason to believe that what can be obtained by two occurrences, A & A, can be obtained by merely one occurence of A. Thus, *there is an element of idealization in intuitionistic logic* and the optional discharge, which allows one to believe that one can obtain something from one use of A, when it is known only that it can be obtained only from two uses of A. In other words, *if a pack of Gauloises costs \$ 2, one cannot buy it with \$ 1*. It should thus be clear that, in cases such as that of allowing the optional discharge, one obtains more than one is strictly entitled to and if one really wishes to follow a radical anti-realist course, such allowances must simply be forbidden.

It turns out that Gentzen's sequent calculi advantageously bring to the fore the rules responsible for these idealizations, i.e., they bring to the fore what is *hidden* within the natural rules. For example, the optional discharge has the structural rule of Thinning on the left as a counterpart in the sequent calculus:

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}$$

Relevant logic and linear logic differ from intuitionistic logic in this that they both reject not just Thinning on the right but also Thinning on the left, so that one does not have $A \rightarrow (B \rightarrow A)$ and $A \rightarrow (\neg A \rightarrow B)$.[11]

Linear logic goes further, however, than relevant logic, as Contraction on both sides is also rejected (more precisely, they are transformed into the 'exponentials' ? and !). Now, inasmuch as the optional discharge has the structural rule of Thinning on the left as a counterpart in the sequent calculus, the obligatory discharge has the rule of Contraction on the left as a counterpart:

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}$$

It is also rejected in linear logic, in which discharge is obligatory but not multiple.

The movement towards a greater restriction of structural rules, which led to the rejection of Thinning and Contraction on both sides within linear logic, seems at first blush nicely to fit the argument for the radicalization of anti-realism that we have sketched, as one can clearly identify the structural rules responsible for the vestiges of Platonism that are still present in intuitionism. Clearly, criteria for assertability modelled on sequent rules in some system of linear logic are desirable from our radical anti-realist perspective. We could not even begin here to discuss this point in any detailed manner, but we hope that this newly established link between radical anti-realism and substructural logics will renew the debate about realism within proof-theoretical semantics.

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5. NOTES

[1] This definition of the realist position, taken from Dummett, implies that a distinction between 'decidable' and 'undecidable' sentences can be drawn. A sentence is *decidable* just in case we have already provided a proof or a refutation of it or we are in possession of an effective method for it, and it is *undecidable* just in case it is not decidable. In other words, a sentence is decidable just in case we know or have reason to think that either we can recognize it to be true or we can recognize it to be false.On the

difficulties linked with this definition, see (Shieh 1998a).

[2] These papers are collected in (Dummett 1978). Dummett has given a definitive statement of his position in *The Logical Basis of Metaphysics* (1991). Among Prawitz's papers, see especially (Prawitz 1977).

[3] See Crispin Wright's paper on 'Strict Finitism' (1993, 107-175) and Neil Tennant's *Anti-Realism and Logic* (1987) and *The Taming of the True* (1997).

[4] The Karp-Cook Thesis originated in (Edmonds 1965) and (Cook 1975), it is stated in (Davis 1982, p. 23). The expression 'Gandy's Thesis' is ours, in memory of the late Robin Gandy, who stated this thesis in 'Limitations to Mathematical Knowledge' (1982, p. 130).

[5] The identification of feasible computing with the Aristotelian notion of infinity comes out quite clearly in (Leivant 1994). Although we wish not to proceed along following Gandy's Thesis, we also have reservations concerning the Karp-Cook Thesis, which is not entirely satisfactory from the radical point of view presented in this paper.

[6] For further improvements on Parikh's result, see (Dragalin 1985) and (Sazonov 1995).

[7] The argument presented in the remainder of this section is spelled out in a much more detailed manner in (Dubucs 1997) to which the reader is referred.

[8] This conception originates in bolzano. See (Dubucs, to appear).

[9] We are leaving aside all exegetical matters. For a more detailed discussion, see (Marion 1998).

[10] Gabriella Crocco (1999) has convincingly argued that the debate about logical revisionism should be framed in this way.

[11] It is worth noticing here that some of the first intuitionistic logicians raised the question of the admissibility of both $A\rightarrow(B\rightarrow A)$ and $A\rightarrow(\neg A\rightarrow B)$. In 1925, Kolmogorov rejected $A\rightarrow(\neg A\rightarrow B)$, which corresponds to Thinning on the right, but did not reject $A\rightarrow(B\rightarrow A)$, i.e. Thinning on the left. In 1929, Glivenko, whose axiomatization was the one used by Gentzen, raised similar points but decided in the end to follow Heyting in keeping both $A\rightarrow(B\rightarrow A)$ and $A\rightarrow(\neg A\rightarrow B)$. These historical remarks are taken from (Dosen 1993). Unsurprisingly, it was another Russian intuitionist, I. E. Orlov, who provided the first axiomatization of relevant logic, in 1928. On this, see (Dosen 1992).

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